The Fifth-order Overtone Vibrations of Quartz Crystal Plates with Higher-order Mindlin Plate Equations

Ji Wang, Lijun Yang, Nisha Sun, Rongxing Wu, Jianke Du, Dejin Huang
Piezoelectric Device Laboratory, School of Engineering, Ningbo University,
818 Fenghua Road, Ningbo, Zhejiang 315211, CHINA
E-mail: {wangji, dujianke, huangdejin}@nbu.edu.cn

***Abstract*—As demands for high frequency quartz crystal resonators rise, we are prompted to design and make overtone devices with the same material, process, and technology. Indeed, high-order overtone resonators have been widely utilized in many applications and further demands in precision products are also growing. The design of overtone resonators and further improvement of the existing ones to meet precision requirements are largely based on empirical approaches, but we found that the technique can be polished with theoretical and analytical efforts as we examine the applications of the Mindlin plate theory in the design and analysis of the fundamental type. Through the extensive improvements of the Mindlin plate theory, we can now analyze the vibration mode couplings, electrode effect, optimal sizes, and thermal behavior, among others. These essential analytical procedures have been implemented in finite element analysis tools with more advanced features such as the nonlinear behavior prediction and eventually circuit parameter extraction. Since it has been proven that the Mindlin plate equations can be used for the vibration analysis of plates at the higher-order overtone modes with accurate prediction of frequency and dispersion relations in the vicinity of cut-off frequencies, we extended the equations to the third-order for the modal behavior and frequency spectra. The results show that earlier knowledge on the proper selection of the sizes of electrode can be proven from our analysis. In addition, the spatial variation and end effects of displacements, particularly of the working mode, can be used in the optimal selection of resonator configuration. The design changes can be used as a way to improve the resonator performance, which has been increasingly degenerating for higher-order overtone types, to meet more stringent requirements. We now extend the plate equations to the fifth-order so the design principle and guidelines can be summarized from more analytical results of overtone vibrations. These predictions on frequency, deformation, and electrode effects from studies with successive orders of equations can be used for resonator design at higher overtone frequencies.**

I. INTRODUCTION

The overtone quartz crystal resonators have been already widely used in various electronic products for their higher vibration frequencies, which is also preferred the fundamental types, while demands for even higher frequencies are growing in emerging applications such as sensors to maintain a broad frequency range for detection and verification. For quartz crystal resonators we have seen products utilizing the 11^{th} overtone or even higher of the thickness-shear mode, and we do not see a limit on the order of overtones we can use. The efforts on overtone products will be continuing as design and manufacturing techniques are improved to make the performance and resonator parameters satisfying requirements for some typical applications in very high frequencies like the atomic clock and radar. For sensor applications, extremely higher-order near 100^{th} has been mentioned and we would like to see what kind performance requirements can be specified for product developments.

As for the analysis and design of overtone resonators, it is preferred that an analytical method similar to that for the fundamental types can be established to provide essential guidelines on the proper selection of blanks, electrodes, and mountings. The proper determination of these parameters, in conjunction with experimental results, will help the optimal design for better performance and, more importantly, reduced the time and cost. The advantage of product design based on analytical approach is evident as fundamental theories such as the Mindlin and Lee plate equations [1-6] have been developed specifically for the high frequency vibrations of quartz crystal plates and numerical methods like the finite element method are being widely adopted in this industry [7-11]. There are apparent difficulties in applying the higher-order plate equations to overtone vibrations at much higher frequencies because the expanded coupling of vibration modes, but the problem itself is well posed and the formulation in the form of governing equations and boundary conditions are straightforward in comparison with fundamental vibrations. The required efforts are in the recapitulation of the plate theory which will include the higher-order modes and necessary correction related to the complicated truncation and simplification schemes [12-13]. This procedure has been demonstrated by our recent efforts on the analysis of the vibrations of the third-order overtone mode of a crystal plate [14-15]. Through the examination of mode couplings, we selected the strongly coupled modes and included them in the plate equations. After a few trials, we obtained the needed

This research is supported by a grant from Zhejiang Provincial Government under the Key Technology Initiative (Project No. 2006C14021) and the National Natural Science Foundation of China (Project No. 10772087).

equations which gave accurate results in the dispersion relations and eventually the frequency spectra, which is the essential reference needed in the resonator design [15]. What we have learned from the process are: 1) we always want to have few vibration modes included in equations so the calculation will be simple, but this may require diligent examination of many coupled vibration modes, 2) coupled equations also need to be corrected and truncated, but the procedure also need to be studied carefully, especially the correction factors, 3) solutions of coupled vibrations of overtone modes can be obtained with familiar method of straight-crested waves also. Our studies show that indeed the third-overtone vibrations exhibit different vibration behavior which is important in the modification of resonator structure from the fundamental types to improve performance [15].

With the third-order overtone equations completed and the solutions obtained, we want to extend the procedure to even higher-order overtone vibrations. The natural step is to study vibrations of the fifth-order overtone thickness-shear mode, but we have to formulate the problem first. Again, we are required to have the fifth-order plate equations which include more coupled vibration modes. In addition, there are more correction factors to be added to the equations. After that, the procedure will be similar as for the analysis of the fundamental vibrations by calculating the dispersion relations to ensure the accuracy of equations and frequency spectra and mode shapes to provide design guidelines.

What we have found from this study is that the fifth-order equations with correction factors we have from earlier studies are not enough to ensure the accuracy of the dispersion relation and frequency spectra. Further efforts are required in finding better correction factors so the dispersion relation will be accurate at cut-off frequencies of the fundamental, third-, and fifth-order thickness-shear modes to ensure, consequently, accuracy of the frequency spectra. The results presented in this study can be used to have a direct review of the complexity of the fifth-order overtone vibrations of crystal plates and related analytical procedure. Of course, the dispersion relation and the frequency spectra also demonstrated the complexity of results and applications.

II. THE FIFTH-ORDER MINDLIN PLATE EQUATIONS

The higher-order Mindlin plate equations are capable to analyze overtone vibrations of piezoelectric plates, as the theory is intended for [1-4]. The difficulties are in the larger number of coupled vibration modes and needed correction factors to ensure the accuracy of equations. The higher-order plate theory has been truncated and corrected in earlier studies [10]. Earlier work on the overtone vibrations of quartz crystal resonators includes the work by Mindlin with the Lee plate theory [6]. As an application, the third-order equations are used for the vibration analysis of the third-order overtone vibrations of a plate in the thickness-shear mode [14-15]. After a few trials, accurate dispersion relation is obtained from the truncated and corrected third-order equations and the frequency spectra and mode shapes are obtained. These results are to be used for the design and analysis of third-order overtone thickness-shear type resonators, because the analysis will provide theoretical results on the effect of structural parameters on essential properties of resonator. This is the same with the design process of the fundamental type of resonator and application is equally or more important due to the sensitivity of device performance regarding to structural parameters. As an extension of earlier studies on the third-order plate equations, we now start from the fifth-order equations with corrections for the analysis.

A. The Fifth-order Mindlin Plate Equatiosn of Motion

The Mindlin plate equations of motion for straight-crested waves of thickness-shear vibration modes up to the fifth-order are [10]

$$
\begin{aligned}
&T_{6,1}^{(0)} = 2b\rho\ddot{u}_2^{(0)} + \frac{2b^3}{3}\rho\ddot{u}_2^{(2)} + \frac{2b^5}{5}\rho\ddot{u}_2^{(4)},\\
&T_{5,1}^{(0)} = 2b\rho\ddot{u}_3^{(0)} + \frac{2b^3}{3}\rho\ddot{u}_3^{(2)} + \frac{2b^5}{5}\rho\ddot{u}_3^{(4)},\\
&T_{1,1}^{(1)} - T_6^{(0)} = \frac{2b^3}{3}\rho\ddot{u}_1^{(1)} + \frac{2b^5}{5}\rho\ddot{u}_1^{(3)} + \frac{2b^7}{7}\rho\ddot{u}_1^{(5)},\\
&T_{6,1}^{(2)} - 2T_2^{(1)} = \frac{2b^3}{3}\rho\ddot{u}_2^{(0)} + \frac{2b^5}{5}\rho\ddot{u}_2^{(2)} + \frac{2b^7}{7}\rho\ddot{u}_2^{(4)},\\
&T_{5,1}^{(2)} - 2T_4^{(1)} = \frac{2b^3}{3}\rho\ddot{u}_3^{(0)} + \frac{2b^5}{5}\rho\ddot{u}_3^{(2)} + \frac{2b^7}{7}\rho\ddot{u}_3^{(4)},\\
&T_{1,1}^{(3)} - 3T_6^{(2)} = \frac{2b^5}{5}\rho\ddot{u}_1^{(1)} + \frac{2b^7}{7}\rho\ddot{u}_1^{(3)} + \frac{2b^9}{9}\rho\ddot{u}_1^{(5)},\\
&T_{6,1}^{(4)} - 4T_2^{(3)} = \frac{2b^5}{5}\rho\ddot{u}_2^{(0)} + \frac{2b^7}{7}\rho\ddot{u}_2^{(2)} + \frac{2b^9}{9}\rho\ddot{u}_2^{(4)},\\
&T_{5,1}^{(4)} - 4T_4^{(3)} = \frac{2b^5}{5}\rho\ddot{u}_3^{(0)} + \frac{2b^7}{7}\rho\ddot{u}_3^{(2)} + \frac{2b^9}{9}\rho\ddot{u}_3^{(4)},\\
&T_{1,1}^{(5)} - 5T_6^{(4)} = \frac{2b^7}{7}\rho\ddot{u}_1^{(1)} + \frac{2b^9}{9}\rho\ddot{u}_1^{(3)} + \frac{2b^{11}}{11}\rho\ddot{u}_1^{(5)},
\end{aligned}
\tag{1}
$$

where variables related to the two-dimensional equations, $T_p^{(n)}(p=1,2,3,4,5,6, n=1,2,3,4,5)$, b, ρ, $u_j^{(n)}(j=1,2,3, n=1,2,3,4,5)$ are abbreviated higher-order stresses, half thickness of plate, density of plate material, and higher-order displacements, respectively.

B. Constitutive Relations

Two-dimensional stresses of AT-cut quartz crystal are given in displacements as

$$
\begin{aligned}
T_5^{(0)} &= 2b\kappa_5^{(0)}\left[c_{55}u_{3,1}^{(0)} + c_{56}\kappa_6^{(0)}\left(u_{2,1}^{(0)} + u_1^{(1)}\right)\right] + \frac{2b^3}{3}\kappa_5^{(2)}\left[c_{55}u_{3,1}^{(2)} + c_{56}\kappa_6^{(2)}\left(u_{2,1}^{(2)} + 3u_1^{(3)}\right)\right]\\
&\quad + \frac{2b^5}{5}\kappa_5^{(4)}\left[c_{55}u_{3,1}^{(4)} + c_{56}\kappa_6^{(4)}\left(u_{2,1}^{(4)} + 5u_1^{(5)}\right)\right],\\
T_6^{(0)} &= 2b\kappa_6^{(0)}\left[c_{65}u_{3,1}^{(0)} + c_{66}\kappa_6^{(0)}\left(u_{2,1}^{(0)} + u_1^{(1)}\right)\right] + \frac{2b^3}{3}\kappa_6^{(2)}\left[c_{65}u_{3,1}^{(2)} + c_{66}\kappa_6^{(2)}\left(u_{2,1}^{(2)} + 3u_1^{(3)}\right)\right]\\
&\quad + \frac{2b^5}{5}\kappa_6^{(4)}\left[c_{65}u_{3,1}^{(4)} + c_{66}\kappa_6^{(4)}\left(u_{2,1}^{(4)} + 5u_1^{(5)}\right)\right],\\
T_1^{(1)} &= \frac{2b^3}{3}\kappa_1^{(1)}\left(c_{11}u_{1,1}^{(1)} + 2c_{12}\kappa_2^{(1)}u_2^{(2)} + 2c_{14}\kappa_4^{(1)}u_3^{(2)}\right) + \frac{2b^5}{5}\kappa_1^{(3)}\left(c_{11}u_{1,1}^{(3)} + 4c_{12}\kappa_2^{(3)}u_2^{(4)}\right)\\
&\quad + \frac{8b^5}{5}\kappa_1^{(3)}c_{14}\kappa_4^{(3)}u_3^{(4)} + \frac{2b^7}{7}\kappa_1^{(5)}c_{11}u_{1,1}^{(5)},\\
T_2^{(1)} &= \frac{2b^3}{3}\kappa_2^{(1)}\left(c_{21}u_{1,1}^{(1)} + 2c_{22}\kappa_2^{(1)}u_2^{(2)} + 2c_{24}\kappa_4^{(1)}u_3^{(2)}\right) + \frac{2b^5}{5}\kappa_2^{(3)}\left(c_{21}u_{1,1}^{(3)} + 4c_{22}\kappa_2^{(3)}u_2^{(4)}\right)\\
&\quad + \frac{8b^5}{5}\kappa_2^{(3)}c_{24}\kappa_4^{(3)}u_3^{(4)} + \frac{2b^7}{7}\kappa_2^{(5)}c_{21}u_{1,1}^{(5)},
\end{aligned}
$$

$$
\begin{aligned}
T_4^{(1)} &= \frac{2b^3}{3}\kappa_4^{(1)}\left(c_{41}u_{1,1}^{(1)} + 2c_{42}\kappa_2^{(1)}u_2^{(2)} + 2c_{44}\kappa_4^{(1)}u_3^{(2)}\right) + \frac{2b^5}{5}\kappa_4^{(3)}\left(c_{41}u_{1,1}^{(3)} + 4c_{42}\kappa_2^{(3)}u_2^{(4)}\right) \\
&\quad + \frac{8b^5}{5}\kappa_4^{(3)}c_{44}\kappa_4^{(3)}u_3^{(4)} + \frac{2b^7}{7}\kappa_4^{(5)}c_{41}u_{1,1}^{(5)}, \\
T_5^{(2)} &= \frac{2b^3}{3}\kappa_5^{(0)}\left[c_{55}u_{3,1}^{(0)} + c_{56}\kappa_6^{(0)}\left(u_{2,1}^{(0)} + u_1^{(1)}\right)\right] + \frac{2b^5}{5}\kappa_5^{(2)}\left[c_{55}u_{3,1}^{(2)} + c_{56}\kappa_6^{(2)}\left(u_{2,1}^{(2)} + 3u_1^{(3)}\right)\right] \\
&\quad + \frac{2b^7}{7}\kappa_5^{(4)}\left[c_{55}u_{3,1}^{(4)} + c_{56}\kappa_6^{(4)}\left(u_{2,1}^{(4)} + 5u_1^{(5)}\right)\right], \\
T_6^{(2)} &= \frac{2b^3}{3}\kappa_6^{(0)}\left[c_{65}u_{3,1}^{(0)} + c_{66}\kappa_6^{(0)}\left(u_{2,1}^{(0)} + u_1^{(1)}\right)\right] + \frac{2b^5}{5}\kappa_6^{(2)}\left[c_{65}u_{3,1}^{(2)} + c_{66}\kappa_6^{(2)}\left(u_{2,1}^{(2)} + 3u_1^{(3)}\right)\right] \\
&\quad + \frac{2b^7}{7}\kappa_6^{(4)}\left[c_{65}u_{3,1}^{(4)} + c_{66}\kappa_6^{(4)}\left(u_{2,1}^{(4)} + 5u_1^{(5)}\right)\right], \\
T_1^{(3)} &= \frac{2b^5}{5}\kappa_1^{(1)}\left(c_{11}u_{1,1}^{(1)} + 2c_{12}\kappa_2^{(1)}u_2^{(2)} + 2c_{14}\kappa_4^{(1)}u_3^{(2)}\right) + \frac{2b^7}{7}\kappa_1^{(3)}\left(c_{11}u_{1,1}^{(3)} + 4c_{12}\kappa_2^{(3)}u_2^{(4)}\right) \\
&\quad + \frac{8b^7}{7}\kappa_1^{(3)}c_{14}\kappa_4^{(3)}u_3^{(4)} + \frac{2b^9}{9}\kappa_1^{(5)}c_{11}u_{1,1}^{(5)}, \\
T_2^{(3)} &= \frac{2b^5}{5}\kappa_2^{(1)}\left(c_{21}u_{1,1}^{(1)} + 2c_{22}\kappa_2^{(1)}u_2^{(2)} + 2c_{24}\kappa_4^{(1)}u_3^{(2)}\right) + \frac{2b^7}{7}\kappa_2^{(3)}\left(c_{21}u_{1,1}^{(3)} + 4c_{22}\kappa_2^{(3)}u_2^{(4)}\right) \\
&\quad + \frac{8b^7}{7}\kappa_2^{(3)}c_{24}\kappa_4^{(3)}u_3^{(4)} + \frac{2b^9}{9}\kappa_2^{(5)}c_{21}u_{1,1}^{(5)}, \\
T_4^{(3)} &= \frac{2b^5}{5}\kappa_4^{(1)}\left(c_{41}u_{1,1}^{(1)} + 2c_{42}\kappa_2^{(1)}u_2^{(2)} + 2c_{44}\kappa_4^{(1)}u_3^{(2)}\right) + \frac{2b^7}{7}\kappa_4^{(3)}\left(c_{41}u_{1,1}^{(3)} + 4c_{42}\kappa_2^{(3)}u_2^{(4)}\right) \\
&\quad + \frac{8b^7}{7}\kappa_4^{(3)}c_{44}\kappa_4^{(3)}u_3^{(4)} + \frac{2b^9}{9}\kappa_4^{(5)}c_{41}u_{1,1}^{(5)}, \\
T_5^{(4)} &= \frac{2b^4}{5}\kappa_5^{(0)}\left[c_{55}u_{3,1}^{(0)} + c_{56}\kappa_6^{(0)}\left(u_{2,1}^{(0)} + u_1^{(1)}\right)\right] + \frac{2b^7}{7}\kappa_5^{(2)}\left[c_{55}u_{3,1}^{(2)} + c_{56}\kappa_6^{(2)}\left(u_{2,1}^{(2)} + 3u_1^{(3)}\right)\right] \\
&\quad + \frac{2b^9}{9}\kappa_5^{(4)}\left[c_{55}u_{3,1}^{(4)} + c_{56}\kappa_6^{(4)}\left(u_{2,1}^{(4)} + 5u_1^{(5)}\right)\right], \\
T_6^{(4)} &= \frac{2b^5}{5}\kappa_6^{(0)}\left[c_{65}u_{3,1}^{(0)} + c_{66}\kappa_6^{(0)}\left(u_{2,1}^{(0)} + u_1^{(1)}\right)\right] + \frac{2b^7}{7}\kappa_6^{(2)}\left[c_{65}u_{3,1}^{(2)} + c_{66}\kappa_6^{(2)}\left(u_{2,1}^{(2)} + 3u_1^{(3)}\right)\right] \\
&\quad + \frac{2b^9}{9}\kappa_6^{(4)}\left[c_{65}u_{3,1}^{(4)} + c_{66}\kappa_6^{(4)}\left(u_{2,1}^{(4)} + 5u_1^{(5)}\right)\right], \\
T_1^{(5)} &= \frac{2b^7}{7}\kappa_1^{(1)}\left(c_{11}u_{1,1}^{(1)} + 2c_{12}\kappa_2^{(1)}u_2^{(2)} + 2c_{14}\kappa_4^{(1)}u_3^{(2)}\right) + \frac{2b^9}{9}\kappa_1^{(3)}\left(c_{11}u_{1,1}^{(3)} + 4c_{12}\kappa_2^{(3)}u_2^{(4)}\right) \\
&\quad + \frac{8b^9}{9}\kappa_1^{(3)}c_{14}\kappa_4^{(3)}u_3^{(4)} + \frac{2b^{11}}{11}\kappa_1^{(5)}c_{11}u_{1,1}^{(5)}.
\end{aligned}
\tag{2}
$$

where $c_{pq}(p,q = 1,2,3,4,5,6)$ and $\kappa_p^{(n)}(p = 1,2,3,4,5,6, n = 1,2,3,4)$ are elastic constants and correction factors [10]. It should be emphasized that the corrections are intended to ensure the cut-off frequencies are exact. The values of these correction factors in this study are

$$
\begin{aligned}
&\kappa_2^{(0)} = \kappa_4^{(0)} = \kappa_6^{(0)} = 0.95236837, \kappa_2^{(1)} = \kappa_4^{(1)} = \kappa_6^{(1)} = 0.88110770, \\
&\kappa_2^{(2)} = \kappa_4^{(2)} = \kappa_6^{(2)} = 0.73240723, \kappa_2^{(3)} = \kappa_4^{(3)} = \kappa_6^{(3)} = 0.78023190, \\
&\kappa_2^{(4)} = \kappa_4^{(4)} = \kappa_6^{(4)} = 0.81748738.
\end{aligned}
\tag{3}
$$

Since the material constants and all parameters in above equations are known, we are ready to study free vibrations of the fifth-order thickness-shear and coupled modes by following earlier procedure for the third-order thickness-shear vibrations.

III. FREE VIBRATIONS OF THE FIFTH-ORDER THICKNESS-SHEAR OVERTONE MODE

It is not surprising that higher-order modes are coupled with lower-order modes in these equations. Further simplification is certainly preferred but the results were disappointing from our earlier experiences. Since it is difficult to examine these equations for possible decoupling in the concerned frequency range, which is the vicinity of the fifth-order thickness-shear in this study, we have to utilize the full set of equations to investigate the overtone vibrations accurately. Following the standard approach with straight-crested waves in a rectangular plate of AT-cut quartz crystal shown in Fig. 1, we assume displacements are in the form of

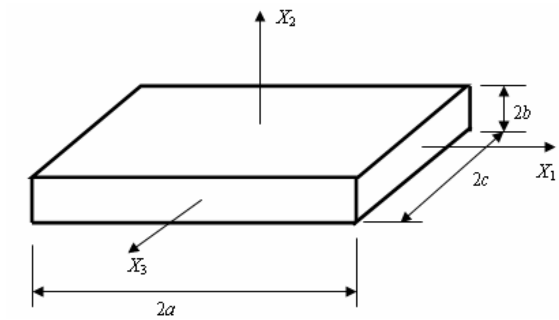

Fig. 1 A typical plate with coordinate system.

$$
\begin{aligned}
&u_2^{(0)} = A_1 \sin\xi x_1 e^{i\omega t}, u_3^{(0)} = A_2 \sin\xi x_1 e^{i\omega t}, u_1^{(1)} = \frac{A_3}{b}\cos\xi x_1 e^{i\omega t}, \\
&u_2^{(2)} = \frac{A_4}{b^2}\sin\xi x_1 e^{i\omega t}, u_3^{(2)} = \frac{A_5}{b^2}\sin\xi x_1 e^{i\omega t}, u_1^{(3)} = \frac{A_6}{b^3}\cos\xi x_1 e^{i\omega t}, \\
&u_2^{(4)} = \frac{A_7}{b^4}\sin\xi x_1 e^{i\omega t}, u_3^{(4)} = \frac{A_8}{b^4}\sin\xi x_1 e^{i\omega t}, u_1^{(5)} = \frac{A_9}{b^5}\cos\xi x_1 e^{i\omega t},
\end{aligned}
\tag{4}
$$

where $A_i(i = 1,2,3,\cdots,9)$, $\xi, x_1, \omega,$ and t are amplitudes, wavenumber, frequency, and time, respectively. The displacements are chosen so that the thickness-shear mode will be symmetric as we expected.

By substituting (4) into stresses in (2) and equations of motion in (1), we can obtain displacements, with time factor omitted, as

$$
\begin{aligned}
&u_2^{(0)} = \sum_{r=1}^{9}\alpha_{1r}A_{9r}\sin\left(\frac{Z_r\pi}{2}\frac{x_1}{b}\right), u_3^{(0)} = \sum_{r=1}^{9}\alpha_{2r}A_{9r}\sin\left(\frac{Z_r\pi}{2}\frac{x_1}{b}\right), \\
&u_1^{(1)} = \sum_{r=1}^{9}\alpha_{3r}\frac{A_{9r}}{b}\cos\left(\frac{Z_r\pi}{2}\frac{x_1}{b}\right), u_2^{(2)} = \sum_{r=1}^{9}\alpha_{4r}\frac{A_{9r}}{b^2}\sin\left(\frac{Z_r\pi}{2}\frac{x_1}{b}\right), \\
&u_3^{(2)} = \sum_{r=1}^{9}\alpha_{5r}\frac{A_{9r}}{b^2}\sin\left(\frac{Z_r\pi}{2}\frac{x_1}{b}\right), u_1^{(3)} = \sum_{r=1}^{9}\alpha_{6r}\frac{A_{9r}}{b^3}\cos\left(\frac{Z_r\pi}{2}\frac{x_1}{b}\right), \\
&u_2^{(4)} = \sum_{r=1}^{9}\alpha_{7r}\frac{A_{9r}}{b^4}\sin\left(\frac{Z_r\pi}{2}\frac{x_1}{b}\right), u_3^{(4)} = \sum_{r=1}^{9}\alpha_{8r}\frac{A_{9r}}{b^4}\sin\left(\frac{Z_r\pi}{2}\frac{x_1}{b}\right), \\
&u_1^{(5)} = \sum_{r=1}^{9}\alpha_{9r}\frac{A_{9r}}{b^5}\cos\left(\frac{Z_r\pi}{2}\frac{x_1}{b}\right),
\end{aligned}
\tag{5}
$$

where

$$
Z_r = \frac{\xi_r}{\frac{\pi}{2b}}, \alpha_{ir} = \frac{A_{ir}}{A_{9r}}, i, r = 1,2,3,4,5,6,7,8,9.
\tag{6}
$$

As a result, stress components in (2) can also be expressed with wavenumber solutions and amplitude ratios of (5) and (6).

To obtain frequency solutions, we need to utilize the traction-free boundary conditions

$$T_5^{(0)} = T_6^{(0)} = T_1^{(1)} = T_5^{(2)} = T_6^{(2)} = T_1^{(3)} = T_5^{(4)} = T_6^{(4)} = T_1^{(5)} = 0 \text{ at } x_1 = \pm a. \tag{7}$$

This results in a frequency equation through setting the coefficient matrix of the boundary condition equations to vanish. The elements of the matrix will not be listed here due to the lengthy expressions, but they are similar to our earlier papers on the third-order plate vibrations in the thickness-shear mode [14-15].

Eventually the displacements are given as

$$\begin{aligned}
u_2^{(0)} &= A_{99}\sum_{r=1}^{9}\beta_r\alpha_{1r}\sin\left(\frac{Z_r\pi}{2}\frac{x_1}{b}\right), u_3^{(0)} = A_{99}\sum_{r=1}^{9}\beta_r\alpha_{2r}\sin\left(\frac{Z_r\pi}{2}\frac{x_1}{b}\right),\\
u_1^{(1)} &= \frac{A_{99}}{b}\sum_{r=1}^{9}\beta_r\alpha_{3r}\cos\left(\frac{Z_r\pi}{2}\frac{x_1}{b}\right), u_2^{(2)} = \frac{A_{99}}{b^2}\sum_{r=1}^{9}\beta_r\alpha_{4r}\sin\left(\frac{Z_r\pi}{2}\frac{x_1}{b}\right),\\
u_3^{(2)} &= \frac{A_{99}}{b^2}\sum_{r=1}^{9}\beta_r\alpha_{5r}\sin\left(\frac{Z_r\pi}{2}\frac{x_1}{b}\right), u_1^{(3)} = \frac{A_{99}}{b^3}\sum_{r=1}^{9}\beta_r\alpha_{6r}\cos\left(\frac{Z_r\pi}{2}\frac{x_1}{b}\right),\\
u_2^{(4)} &= \frac{A_{99}}{b^4}\sum_{r=1}^{9}\beta_r\alpha_{7r}\sin\left(\frac{Z_r\pi}{2}\frac{x_1}{b}\right), u_3^{(4)} = \frac{A_{99}}{b^4}\sum_{r=1}^{9}\beta_r\alpha_{8r}\sin\left(\frac{Z_r\pi}{2}\frac{x_1}{b}\right),\\
u_1^{(5)} &= \frac{A_{99}}{b^5}\sum_{r=1}^{9}\beta_r\alpha_{9r}\cos\left(\frac{Z_r\pi}{2}\frac{x_1}{b}\right),\\
\beta_i &= \frac{A_{9i}}{A_{99}}, i = 1,2,3,4,5,6,7,8.
\end{aligned} \tag{8}$$

With the rectangular plate of quartz crystal shown in Fig. 1, we use above equations and procedure for the analysis of the dispersion relation and frequency spectra in the vicinity of the fifth-order thickness-shear overtone mode. From computation, we have the dispersion relations from the coupled vibrations in Fig. 2. It shows that cut-off frequencies are slightly off from the exact values, even though correction factors have been inserted with the objective to make them exact. Such results are to be improved with new correction factors and modified equations.

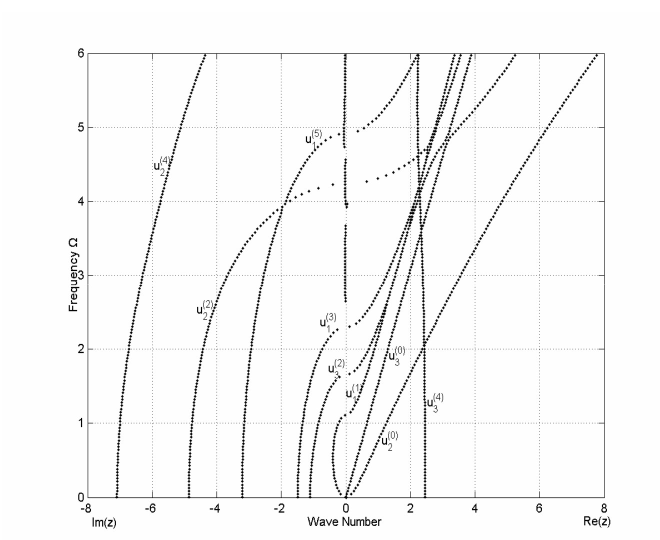

Fig. 2 Dispersion relation of the fifth-order overtone thickness-shear vibrations of a plate

Although the inaccuracy of the dispersion relation usually means that equations have to be reexamined before further calculations can be carried out, we have calculated the frequency spectra with the complete program by following the standard procedure. The frequency spectra in the vicinity of the fifth-order thickness-shear overtone vibrations are given in Fig. 3. The results, as expected, further showed the erroneous behavior of the frequency dependence on aspect ratios of the plate. Many of branches, although coupled, are not in the way of change we are expecting. It further shows that either modification of the coupled equations or the new correction factors are needed.

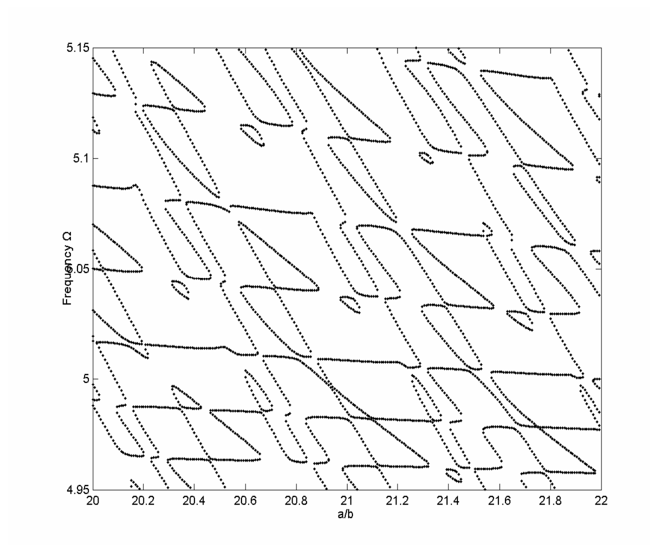

Fig. 3 Frequency spectra of a rectangular AT-cut quartz crystal plate vibrating in the vicinity of the fifth-order thickness-shear overtone mode

The procedure and results presented in this section are the same from the third-order overtone vibrations of a rectangular quartz crystal plate vibrating in the overtone thickness-shear mode. The dispersion relation and frequency spectra, however, shows that the fifth-order equations with correction factors given in (3) and modes considered are not enough to give accurate results we are looking for. This is exactly what happened to the third-order equations, and the remedy comes from the modified equations [14-15]. In this case, we may need to look into the correction factors also because the coupling is more complicated than the way the correction factors are determined.

IV. CONCLUSIONS

We have utilized the fifth-order Mindlin plate equations with correction factors for the analysis of the fifth-order overtone thickness-shear vibrations of a rectangular quartz crystal plate. The procedure has been established for the calculation of dispersion relations, frequency spectra, and selected vibration modes. Such an analysis of the third-order overtone of thickness-shear vibrations has been successful and important findings on the frequency variation and mode shape changes have been obtained. In the case of the fifth-order vibrations, we have found that the dispersion relations and consequently the frequency spectra are not accurate enough. This reminds us that either the equations are to be modified with more couplings or the correction factors have to be modified with the consideration of couplings. We expect further efforts in the revision of the equations and correction factors will result in accurate dispersion relations and consequently frequency spectra. Eventually, such detailed studies on the successive overtone vibrations in the thickness-shear modes will provide important design guidelines for overtone quartz crystal resonators.

ACKNOWLEDGMENT

The authors are grateful for additional financial support from the K. C. Wong Magana Fund established by the K. C. Wong Foundation and administered by Ningbo University.

REFERENCES

[1] R. D. Mindlin (edited by Jiashi Yang), *An Introduction to the Mathematical Theory of Vibrations of Elastic Plates*. World Scientific, New Jersey, 2007.

[2] R. D. Mindlin, Wave and vibrations in isotropic, elastic plates. *Proceedings of the First Symposium on Naval Structural Mechanics*, Pergamon Press, New York, 199-232, 1960.

[3] R. D. Mindlin, High frequency vibrations of crystal plates. *Quarterly of Applied Mathematics*, XIX(1), 51-61, 1961.

[4] R. D. Mindlin, High frequency vibrations of piezoelectric crystal plates. *Inlernational Journal of Solids Struction*, 8, 895-906, 1972.

[5] R. D. Mindlin, Third-order overtone quartz resoantor. *Inlernational Journal of Solids Struction*, 18 (9), 809-817, 1982.

[6] P. C. Y. Lee, J. D. Yu, and W.-S. Lin, A new two-dimensional theory for vibrations of piezoelectric crystal plates with electroded faces. *J. Appl. Phys*., 83 (3), 1213-1223, 1998.

[7] Yook-Kong Yong and Zheng Zhang, A perturbation method for the finite element modeling of piezoelectric vibrations in quartz crystal resonators. *IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control,* 40(5), 551-562, 1993.

[8] Yook-Kong Yong, Ji Wang, and T. Imai, On the accuracy of Mindlin plate predictions for the frequency-temperature behavior of reonant modes in AT- and SC-cut quartz plates. *IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control,* 46(1), 1-13, 1999.

[9] Ji Wang, Yook-Kong Yong, and T. Imai, Finite element analysis of the piezoelectric vibrations of quartz plate resonators with higher-order plate theory. *International Journal of Solids and Structures*, 36(15), 2303-2319, 1999.

[10] Ji Wang, J-D Yu, Y-K Yong, and T. Imai, A new theory for electroded piezoelectric plates and its finite element application for the forced vibration of quartz crystal resonators. *International Journal of Solids and Structures*, 37, 5653-5673, 2000.

[11] Ji Wang, Jiun-Der Yu, Yook-Kong Yong, and Tsutomu Imai: A Finite Element Analysis of Frequency-temperature Relations of AT-cut Quartz Crystal Resonators with Higher-order Mindlin Plate Theory. *Acta Mechanica*, 199 (1-4), 117-130, 2008. [doi: 10.1007/s00707-007-0538-5]

[12] Ji Wang, J-D Yu, and Y-K Yong, On the correction of higher-order Mindlin plate theory. *International Journal of Applied Electromagnetics and Mechanics*, 22, 83-96, 2005.

[13] Ji Wang, Rongxing Wu, Liming Gao, and Jianke Du, Correction facetors for Mindlin high-order plate theory with the consideration of electrodes. *Proceedings of the IEEE 2007 International Frequency Control Symposium*, 203-207, 2007.

[14] Ji Wang, Rongxing Wu, Jiaming Xu, Xiaofeng Zheng, Jianke Du, Min-Chiang Chao, and Dejin Huang, The analysis of thickness-shear vibrations of the third-order overtone mode of quartz crystal plates with Mindlin plate theory. *Proceedings of the IEEE 2007 International Ultrasonics Symposium*, 138-141, 2007.

[15] Ji Wang, Rongxing Wu, Jianke Du, and Huiming Wang: The Analysis of the Third-order Thickness-shear Overtone Vibrations of Quartz Crystal Plates with Mindlin Plate Theory, Proceedings of the 2008 IEEE International Ultrasonics Symposium, 2173-2176, 2008. [doi: 10.1109/ULTSYM.2008.0538]